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RESEARCH ARTICLE

Robust portfolio optimization with a generalized expected utility model under ambiguity

Xiaoxian Ma · Qingzhen Zhao · Jilin Qu

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Abstract This paper proposes a robust approach maximizing worst-case utility when both the distributions underlying the uncertain vector of returns are exactly unknown and the estimates of the structure of returns are unreliable. We introduce concave convex utility function measuring the utility of investors under model uncertainty and uncertainty structure describing the moments of returns and all possible distributions and show that the robust portfolio optimization problem corresponding to the uncertainty structure can be reformulated as a parametric quadratic programming problem, enabling to obtain explicit formula solutions, an efficient frontier and equilibrium price system.

Keywords Robust optimization · Portfolio selection · Ambiguity · Equilibrium

JEL Classification C02 · C44 · D81

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X. Ma (✉)
College of Management and Economics, Shandong Normal University,
School of Finance and Banking, Shandong University of Finance, Jinan 250014, China
e-mail: xiaoxianma@tom.com

Q. Zhao
College of Management and Economics, Shandong Normal University, Jinan 250014, China

J. Qu
School of Accounting, Shandong University of Finance, Jinan 250014, China

1 Introduction

Expected utility maximization and return-risk trade-off are two widely accepted types of frameworks in financial optimization for modeling decision making under uncertainty. The former is mainly used in theoretical study, while the latter is widely applied both in theoretical study and in practice. In this paper, we will transform a portfolio optimization model with utility maximization into one with return-risk trade-off.

Portfolio optimization is one of the best known and most widely used methods in portfolio selection. The first mathematical model for portfolio selection was formulated by Markowitz (1952). In mean-variance portfolio selection model, the return on a portfolio is measured by the expected value of the portfolio return, and the associated risk is quantified by the variance of the portfolio return. This mean-variance model has had a profound impact on the economic modeling of financial markets and the pricing of assets. However, as pointed by Black and Litterman (1992) and Michaud (1998), despite the elegance of the mean-variance model, the powerful optimization theory supporting this model, and the availability of efficient software to solve the resulting problem, the mean-variance optimization has constantly encountered skepticism among investment practitioners. One of the reasons for its being skepticized is that optimal portfolios are often sensitive to changes in input parameters, leading to large turnover ratios with periodic readjustment of input estimates such as the mean and covariance matrix, especially the mean (Merton 1980; Broadie 1993; Chopra 1993). Robust portfolio optimization can offer vehicles to incorporate estimation risk into the decision making process in portfolio selection (see for example, Halldórsson and Tütüncü 2003; Goldfarb and Iyengar 2003; Gundel 2005; Bertsimas and Brown 2005).

Expected utility theory (von Neumann and Morgenstern 1944) reigned for several decades as the dominant normative and descriptive model of decision making under uncertainty, but it does not provide an adequate description of individual choice. Several generalized models of the expected utility theory were inspired by experimental evidence and aimed at improving its descriptive accuracy, including the anticipated utility model of Quiggin (1982), the dual theory of choice model due to Yaari (1987) and the cumulative prospect theory model of Tversky and Kahneman (1992). Recently, some researches of cumulative prospect theory were reported in Basili et al. (2005), Stott (2006) and Rieger and Wang (2006), etc.

Ambiguity, also called model uncertainty, has recently become a research topic of interest in the portfolio optimization problem to obtain its robust solution. Epstein and several coauthors (see Epstein and Wang 1994; Epstein and Schneider 2003; Epstein and Schneider 2006, for example) formulated recursive multi-priors utility model and showed that it afforded the Knightian distinction between risk and ambiguity and, Hansen, Sargent and several coauthors proposed a related nonaxiomatic model based on robust control theory (see Anderson et al. (2000), for example). Both streams of the literature have as their origin the multi-priors model in Gilboa and Schmeidler (1989) (Chen and Epstein 2002). In addition, Benartzi and Thaler (1995) and Barberis et al. (2001) studied how loss aversion affected portfolio decisions. Wang (2005) proposed a shrinkage approach to investigate the implications of aversion to model uncertainty. Garlappi et al. (2006) gave a multi-prior approach for portfolio selection with parameter and model uncertainty. Nau (2006) presented a axiomatic model of nonneutral attitudes toward uncertainty. Schied (2007) showed that investor preferences under aversion against both risk and ambiguity can be numerically represented in terms of convex risk measures using a duality approach.

With no explicit state space, portfolio optimization problems can be analysed by moment conditions. Generalized method of moments (Hansen 1982) provided an attractive estimation methodology that has been widely used in financial research and is well suited to situations where market information is given in terms of moment conditions. It is complementary to maximum likelihood method and its Bayesian counterpart. Literature on asset market pricing and portfolio selection with moments estimation include Rubinstein (1973), Bansal et al. (1993), Chapman (1997), Harvey and Siddique (2000), Chernov et al. (2003), Athayde and Flôres (2004), Kitamura and Otsu (2005), Singleton (2006), Guidolin and Timmermann (2006) and Cvitanic et al. (2007), etc.

Gilboa and Schmeidler (1989) and Schmeidler (1989) formulated axioms on investor preferences that should account for aversion against both risk and ambiguity. They showed that these preferences can be numerically represented by a coherent robust utility function. Recently, there has been considerable interest in studying optimization problems in which the target function is defined in terms of a coherent or convex risk measure, see Talay and Zheng (2002), Föllmer and Schied (2004) and Quenez (2004), for example. These optimization problems can be called robust since optimization involves an entire class of possible probabilistic models and thus takes into account model risk.

Our model in this paper can be applied to a market environment in which not only the distributions of return variables unknown exactly but also the estimates of the market parameters subjected to statistical errors. This model of utility is an extension of Gilboa-Schmeidler multiple priors model where the single probability measure of the standard Savage model is replaced by a set of probabilities or priors.

When the utility function is quadratic or linear, expected utility optimization model will not depend on the actual distribution of the return variable, except its mean and covariance. Therefore, many elegance properties of utility maximization with quadratic utility function are obtained. Using general utility function having one or two support points, Popescu (2007) proposed an approach for deriving robust solutions for stochastic optimization programs, based on mean-covariance information of the distributions underlying the return vector. Further, few papers are reported when estimates of the parameters of random return variable are unreliable.

In this paper we will consider the portfolio problem for the case when the moments of the multivariate distribution of returns are not known exactly and present a solution of the problem using a generalized expected utility model in combination with an inverse S-shaped probability transformation function. The remainder of this paper is organized as follows. In Sect. 2, an optimal portfolio selection problem and a concave convex utility function are stated and a MaxMin model is constructed. In Sect. 3, we transform the maximization utility model to a return-risk trade-off model and obtain the explicit formula solutions and efficient frontier. In Sect. 4, explicit formula for equilibrium price system is given. We present our conclusions in Sect. 5.

2 Problem statement and model construction

In this paper, we consider a capital market with n risky assets offering random rates of returns, a risk-less asset offering a fixed rate of return and $m(< n)$ factors that drive the market. The returns on assets in different market periods are assumed to be independent. The distribution of the uncertain returns is not exactly known and only partial information about this distribution is available or reliable, such as variances and co-variances, and expected means of returns are assumed to vary over an interval. An investor allocates his wealth among the n risky assets and the risk-less asset, and is interested in maximizing his expected utility under the worst case. As usual, we assume that no costs and no taxes are associated with transactions, all assets are infinitely divisible and have no limitation on short-sales, and the total value of the assets as well as the total number of shares of each asset remain constant during the transaction.

The following notations will be employed:

r_i the holding period rate of return on risky asset i $(i = 1, 2, \dots, n)$, which equals to the ratio of the value of the asset at the end of the period over its current value, $\mathbf{r} = (r_1, r_2, \dots, r_n)^T$,

μ_i the expected value of r_i $(i = 1, 2, \dots, n)$, $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n)^T$,

r_f the holding period rate of return on the risk-less asset $(n+1)$, which is a constant,

f_i the holding period rate of return on factor, $\mathbf{f} = (f_1, f_2, \dots, f_m)^T$,

$\mathbf{F}$ the covariance matrix of factor return vector $\mathbf{f}$,

σ_{ij} the factor loading f_i and r_j, $i = 1, 2, \dots, m,\ \ j = 1, 2, \dots, n$, $\boldsymbol{\sigma} := (\sigma_{ij})_{m\times n}$ is the factor loading matrix,

ε_i the holding period residual return, $i = 1, 2, \dots, n$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)^T$,

x_i the proportion of the wealth invested in asset i owned by the investor after the transaction, $i = 1, 2, \dots, n$, $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$, x_{n+1} is the proportion of the wealth invested in risk-less asset,

p_i the price of per share asset i, $i = 1, 2, \dots, n + 1$; $p_{n+1} = 1$,

y_i the number of shares of asset i owned by the investor after the transaction, $i = 1, 2, \dots, n + 1$,

y_i^0 the number of shares of asset i owned by the investor before the transaction, $i = 1, 2, \dots, n + 1$,

$\mathbf{D}$ the residual covariance matrix, $\mathbf{V} = \boldsymbol{\sigma}^T\mathbf{F}\boldsymbol{\sigma} + \mathbf{D}$, $\mathbf{V_x} = \mathbf{x}^T\mathbf{V}\mathbf{x}$ and $\boldsymbol{\mu}_\mathbf{x} = \mathbf{x}^T\boldsymbol{\mu} + x_{n+1}r_f$,

$\mathbf{r} \sim (\boldsymbol{\mu}, \mathbf{V})$ the probability distribution of $\mathbf{r}$ belongs in a class $\mathrm{M}^n_{(\boldsymbol{\mu},\mathbf{V})}$ of n-variant distributions with mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{V}$.

Throughout the paper, vectors and matrixes will be denoted in bold, to be distinguished from other variables.

We assume that $\mu_i \geq r_f$, $i = 1, 2, \dots, n$, which means that the expected rate of return on each risky asset is not less than that on the risk-less asset (if such a risky asset do exist in the original problem, the investor would not incorporate this risky asset into his portfolios as a best investment strategy).

Consider a robust factor model for the assets returns. In this model the vector of random asset returns of single period is given by

$$\mathbf{r} = \boldsymbol{\mu} + \boldsymbol{\sigma}^T\mathbf{f} + \boldsymbol{\varepsilon}, \tag{2.1}$$

Where $\mu \in \mathbf{R}^n$ is the vector of mean returns, $\sigma \in \mathbf{R}^{m\times n}$ is the matrix of factor loadings of the n assets, $\mathbf{f}$ is the vector of returns of the factors that drive the market subject to $\mathbf{f} \sim (\mathbf{0}, \mathbf{F}) \in \mathbf{R}^m$ and ε is the vector of residual returns subject to $\varepsilon \sim (\mathbf{0}, \mathbf{D})$.

As usual, throughout the paper we assume that the vector of residual returns ε is independent of the vector of factor returns $\mathbf{f}$, the covariance matrix $\mathbf{F}$ is positive definite and the diagonal covariance matrix $\mathbf{D} = diag(\mathbf{d})$ is nonnegative definite, i.e., $d_i \geq 0,\ i = 1, 2, \ldots, n$. Thus, the vector of asset returns $\mathbf{r} \sim (\mu, \mathbf{V})$ is a n- variant distribution. In practice, the eigenvalues of the residual covariance matrix $\mathbf{D}$ are typically much smaller than those of the covariance matrix $\sigma^T \mathbf{F} \sigma$ implied by the factors; i.e., matrix $\sigma^T \mathbf{F} \sigma$ is a good low-rank approximation of the covariance of the asset returns. Matrix $\mathbf{V}$ is positive definite when the matrix of factor loadings is full-rank or the residual covariance matrix is positive definite. Therefore, the investor can acquire a positive definite matrix $\mathbf{V}$ by adjusting target assets or giving matrix $\mathbf{D}$ a small positive perturbation.

The rate of random return of portfolio $(x_1, x_2, \ldots, x_n, x_{n+1})^T$ is given by

$$r_{\mathbf{x}} = x_{n+1} r_f + \mathbf{x}^T \mu + \mathbf{x}^T \sigma^T \mathbf{f} + \mathbf{x}^T \varepsilon. \tag{2.2}$$

In the event that the vector of residual returns ε is independent of the vector of factor returns $\mathbf{f}$, $r_{\mathbf{x}}$ is a projected univariant distribution subject to $r_{\mathbf{x}} \sim (\mu_{\mathbf{x}}, \mathbf{V}_{\mathbf{x}})$.

The covariance matrix $\mathbf{F}$ of the factor return vector $\mathbf{f}$ is assumed to be stable and known exactly. But the mean vector μ, factor loading matrix σ, and the residual error vector ε in our model are all known to lie within suitably defined uncertainty sets.

The mean returns vector μ is assumed to lie in the uncertainty set S_μ given by

$$S_\mu = \left\{\mu \,\middle|\, \mu = \mu_0 + diag(\theta_1)\gamma,\ \gamma = (\gamma_1, \gamma_2, \ldots, \lambda_n)^T, \theta_1 \in (0, 1)^n\right\}, \tag{2.3}$$

i.e., each component of μ is assumed to lie within a certain interval.

The columns of the matrix σ, i.e., the factor loadings of the individual assets, are also assumed to known approximately. In particular, σ belongs to the uncertainty set S_σ given by

$$\begin{aligned} S_\sigma = \Big\{\sigma \,\Big|\, \sigma_j = \sigma_{0j} + \theta_{2j}\rho_j,\ \rho_j = (\rho_{1j}, \rho_{2j}, \ldots, \rho_{mj})^T, \\ i = 1, 2, \ldots, m,\ \theta_2 \in (0, 1)^n\Big\}. \end{aligned} \tag{2.4}$$

The individual diagonal elements of the residual covariance matrix are assumed to lie in an interval, i.e., the uncertainty set $S_{\mathbf{d}}$ for the matrix $\mathbf{D}$ is given by

$$S_{\mathbf{d}} = \left\{D \,\middle|\, D = D_0 + diag(\theta_3) diag(\eta),\ \eta = (\eta_1, \eta_2, \ldots, \eta_n), \theta_3 \in (0, 1)^n\right\}. \tag{2.5}$$

Let $\theta = (\theta_1, \theta_2, \theta_3)^T, \theta \in R^{3\times n}$. The reasons for setting conditions (2.3)–(2.5) are, as described in the introduction, that the values of the means and covariance of returns on risky assets can not be estimated exactly and the composition of the optimal portfolio can be extremely sensitive to estimation errors in the asset means and variances among problem inputs. The lower bounds $\boldsymbol{\mu}_0, \boldsymbol{\sigma}_0, \mathbf{D}_0$ and upper bounds $\bar{\boldsymbol{\mu}} = \boldsymbol{\mu}_0 + \boldsymbol{\gamma}, \bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}_0 + \boldsymbol{\rho}, \bar{\mathbf{D}} = \mathbf{D}_0 + diag(\boldsymbol{\eta})$ may also be estimated by linear regression, Bayesian or statistic in historical data, in analysts' estimates, in simulated scenarios, etc. See Goldfarb and Iyengar (2003) for some detailed discussion about how to determine these bounds.

The uncertainty set Π is given by

$$\Pi = \left\{(\boldsymbol{\mu}, \boldsymbol{\sigma}, \mathbf{D}) \,\middle|\, \boldsymbol{\mu} \in S_{\boldsymbol{\mu}},\ \boldsymbol{\sigma} \in S_{\boldsymbol{\sigma}},\ \mathbf{D} \in S_{\mathbf{d}}\right\}. \tag{2.6}$$

An uncertainty structure is denoted with the uncertainty set and all possible distributions which moments of returns belong to above uncertainty set.

An investor's position in this market is described by a portfolio. The budget constraint is given by

$$\mathbf{X} = \left\{(x_1, x_2, \ldots, x_{n+1}) \,\middle|\, \sum_{i=1}^{n+1} x_i = 1\right\}. \tag{2.7}$$

The decision vector represents a portfolio financial assets in the sense that x_i is the position in asset i. The optimization problem will be subject to this portfolio constraint.

Generally, utility function is assumed to be a concave function in classical expected utility theory in which the utility is increasing and the marginal utility diminishing along with the wealth increasing. Prospect theory has suggested the very opposite of classical economics for losses, that is, utility is convex rather than concave. Empirically, concave utility has indeed been well established for gains, but measurements of the utility for losses show mixed results but most evidence supports convex rather than concave utilities (Fennema and Assen 1999). In this paper, we consider the market environment in which an investor's utility is utility for gains (concave function) about random return and utility for losses (convex function) about model uncertainty factor θ. The investor's utility function can be deemed to a combine of a concave utility function and a convex uncertainty factor function which adjusts the degree of uncertainty.

Our investigation focuses on the problem that maximizing the worst expected utility over all possible distributions with any element belonging to the uncertainty set defined in (2.6). The optimization problem can be formulated as follows:

$$\underset{\mathbf{x}\in\mathbf{X}}{Max} \quad \underset{(\boldsymbol{\mu},\boldsymbol{\sigma},\mathbf{D})\in\Pi}{Min} \quad \underset{\mathbf{r}\sim(\boldsymbol{\mu},\mathbf{V})}{Min}\ E[U_\theta(r_\mathbf{x})], \tag{2.8}$$

where the function $U_\theta(r_\mathbf{x})$ is the measure of utility of the investor, θ is a parameter matrix called uncertainty factor that control the distributions of the factor return, the residual return, and the factor loading matrix.

For a given portfolio $\mathbf{x}$ and an element $(\boldsymbol{\mu}, \boldsymbol{\sigma}, \mathbf{D})$ belonging to the uncertainty set Π, we refer to problem

$$\psi_\theta(\mathbf{x}) = \underset{\mathbf{r}\sim(\boldsymbol{\mu},\mathbf{V})}{Min} \quad E[U_\theta(r_\mathbf{x})] \tag{2.9}$$

as the inner optimization problem and to function $\psi_\theta(\mathbf{x})$ as the robust objective.

3 The problem transform and efficient frontier

A real function f defined on a convex set $\mathbf{C}$ is said to be quasi-concave if and only if $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq \min(f(\mathbf{x}), f(\mathbf{y}))$ for any $\lambda \in (0, 1)$ and $\mathbf{x}, \mathbf{y} \in \mathbf{C}$. Function f is said to be quasi-convex if $-f$ is a quasi-concave function. A function f defined on R is convex-concave if there exists $x_0 \in R$ such that f is convex on $(-\infty, x_0)$ and concave on (x_0, ∞). A function defined on R is concave–convex if $-f$ is convex–concave. A function is S-shaped if it is increasing and convex–concave. A function f is inverse S-shaped if $-f$ is S-shaped.

Now we quote a lemma serving as a key to transform the optimization problem (2.8) into a tractable one.

Lemma 3.1 (Popescu 2007) *For any fixed uncertainty factor, if the utility function $U_\theta(r_\mathbf{x})$ is twice differentiable with a monotonic second derivative, then we have*

$$\psi_\theta(\mathbf{x}) = \mathop{Min}_{p\in(0,1)} U_\theta(p, \mathbf{x}); \tag{3.1}$$

if the utility function is differentiable such that U' is inverse S-shaped with finite limits at $\pm\infty$, then we have

$$\psi_\theta(\mathbf{x}) \equiv K_\theta(\mu_x, \mathbf{V}_x) = \mathop{Min}_{p\in(0,1)} K_\theta(p, \mu_x, \mathbf{V}_x), \tag{3.2}$$

$$\begin{aligned} K_\theta\left(p, \mu_x, \mathbf{V}_x\right) = p \cdot U_\theta\left(\mu_x + \sqrt{\frac{(1-p)\mathbf{V}_x}{p}}\right) \\ +(1-p) \cdot U_\theta\left(\mu_x - \sqrt{\frac{p\mathbf{V}_x}{1-p}}\right). \end{aligned} \tag{3.3}$$

The functions satisfying these conditions of Lemma 3.1 include combinations of exponentials and logarithms, for example, log-logistic $U(x) = C + \log 1/(1 + \exp(-ax))$ used in statistics, classification and include hyperbolic functions, for example, the catenary $U(x) = C - b\cosh(ax)$ used in physics, engineering, where in all cases the constant C can be replaced by a concave quadratic (Popescu 2007).

Proposition 3.1 *The inner robust objective function $\psi_\theta(\mathbf{x})$ is a concave function about the portfolio $\mathbf{x}$ and a convex function about uncertainty factor θ under the conditions of Lemma 3.1 and $U_\theta(\mathbf{x})$ is a continuous function.*

Proof In view of the formulas (3.1)–(3.3) in Lemma 3.1 and concavity of utility function, function $\psi_\theta(\mathbf{x})$ is concave about $\mathbf{x}$ since both a convex combination of concave functions and a minimum of a set of concave functions is also a concave function. $\psi_\theta(\mathbf{x})$ is convex function about uncertainty factor θ by the assumption of utility function and formulas (3.1)–(3.3). This completes the proof.

The most popular way of solving a MaxMin problem is transforming it into a MinMax problem.

Lemma 3.2 (Fan 1953) *Let* $\mathbf{X}$ *be a nonempty set and* $\mathbf{Y}$ *be nonempty compact topological space. Let* $\psi : \mathbf{X} \times \mathbf{Y} \to R$ *be lower semi-continuous on* $\mathbf{Y}$*. Suppose that* ψ *is weakly convex on* $\mathbf{X}$ *and weakly concave on* $\mathbf{Y}$*, that is to say: for any* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{X}$ *and* $\alpha \in [0, 1]$*, there exists* $\mathbf{x}_3 \in \mathbf{X}$ *such that*

$$\psi(\mathbf{x}_3, \mathbf{y}) \leq \alpha\psi(\mathbf{x}_1, \mathbf{y}) + (1-\alpha)\psi(\mathbf{x}_2, \mathbf{y}),\ \ \forall \mathbf{y} \in \mathbf{Y},$$

and for any $\mathbf{y}_1, \mathbf{y}_2 \in \mathbf{Y}$ *and* $\beta \in [0, 1]$*, there exists* $\mathbf{y}_3 \in \mathbf{Y}$ *such that*

$$\psi(\mathbf{x}, \mathbf{y}_3) \geq \beta\psi(\mathbf{x}, \mathbf{y}_1) + (1-\beta)\psi(\mathbf{x}, \mathbf{y}_2),\ \ \forall \mathbf{x} \in \mathbf{X}.$$

Then we have

$$\underset{\mathbf{x}\in\mathbf{X}}{Min}\,\underset{\mathbf{y}\in\mathbf{Y}}{Max}\,\psi(\mathbf{x}, \mathbf{y}) = \underset{\mathbf{y}\in\mathbf{Y}}{Max}\,\underset{\mathbf{x}\in\mathbf{X}}{Min}\,\psi(\mathbf{x}, \mathbf{y}). \tag{3.4}$$

It is clear that, in problem (2.8), $\mathbf{X}$ *and* Π *are nonempty compact sets, therefore by Proposition 3.1 and Lemma 3.2, we have the following result.*

Proposition 3.2 *If the utility function is continuous and satisfies that: it is twice differentiable with a monotonic second derivative; or if the utility function is differentiable such that* U' *is inverse S-shaped with finite limits at* $\pm\infty$*, then we have problem (2.8) is equivalent to problem*

$$\underset{(\boldsymbol{\mu},\boldsymbol{\sigma},\mathbf{D})\in\Pi}{Min}\ \ \underset{\mathbf{x}\in\mathbf{X}}{Max}\ \psi_{\Theta}(\mathbf{x}) \tag{3.5}$$

The following result provides general sufficient conditions for solving bicriteria problems efficiently as parametric programs:

Lemma 3.3 (Geoffrion 1967) *Consider the program*

$$\underset{\mathbf{x}\in\mathbf{X}}{Max}\ v(g(f_1(\mathbf{x})), h(f_2(\mathbf{x}))), \tag{3.6}$$

where $\mathbf{X} \in R^n$ *is a convex set. Suppose that* $f_1, f_2, g(f_1), h(f_2)$ *are concave on* $\mathbf{X}$*,* g *and* h *are increasing on the image of* $\mathbf{X}$ *under* f_1, f_2 *respectively, and* v *is non-decreasing in each coordinate and quasi-concave on the convex hull of the image of* $\mathbf{X}$ *under* $(g(f)_1, h(f_2))$*, and all functions are continuous. Further assume that the optimal solution* $\mathbf{x}^*(\lambda)$ *of the parametric program* $\underset{\mathbf{x}\in\mathbf{X}}{Max}\,\lambda f_1(\mathbf{x}) + (1-\lambda)f_2(\mathbf{x})$ *is continuous for* $\lambda \in [0, 1]$*. Then the function* $w(\lambda) = v(g(f_1(\mathbf{x}^*(\lambda))), h(f_2(\mathbf{x}^*(\lambda))))$ *is continuous and unimodal on* $[0, 1]$*. Furthermore, if* λ^* *is its maximum, then* $\mathbf{x}^*(\lambda^*)$ *is optimal for (3.6).*

Based on Lemma 3.3, the following result shows that problem

$$\underset{\mathbf{x}\in\mathbf{X}}{Max}\ \psi_{\Theta}(\mathbf{x}) \tag{3.7}$$

can be optimized by solving a parametric quadratic program.

Proposition 3.3 *Under the conditions of Proposition 3.2, problem (3.7) is equivalent to solving the parametric quadratic program:*

$$\underset{\mathbf{x}\in\mathbf{X}}{Max}\,\lambda(\mathbf{x}^T\mu + x_{n+1}r_f) - (1-\lambda)\,\mathbf{x}^T\mathbf{V}\mathbf{x}. \tag{3.8}$$

Proof Let $f_1 = \mathbf{x}^T\mu + x_{n+1}r_f$ and $f_2 = -\mathbf{x}^T\mathbf{V}\mathbf{x}$, $g(f_1) = f_1$ and $h(f_2) = -\sqrt{(-f_2)}$, these are all concave, and the latter two are non-decreasing. In the notation of Lemma 3.3, we obtain $v(\mu_x, -\mathbf{V}_x) = K_\theta(\mu_x, \mathbf{V}_x)$ for any θ. In view of formulas (3.3) and (3.2), monotonicity of U implies that $K_\theta(p, \mu_x, \mathbf{V}_x)$ is non-decreasing in μ_x. Concavity of U implies monotonicity of $K_\theta(p, \mu_x, \mathbf{V}_x)$ in $\mathbf{V}_\mathbf{x}$ and concavity in $(\mu_x, \mathbf{V}_x)$. Since monotonicity and concavity are preserved through minimization, the same results hold for $K_\theta(\mu_x, \mathbf{V}_x)$. This completes the proof.

The constrained optimization problem (3.8) can be formulated as the following unconstrained optimization problem.

$$\underset{\mathbf{x}\in R^n}{Max}\,\lambda r_f + \lambda\mathbf{x}^T(\mu - r_f\mathbf{I}_n) - (1-\lambda)\,\mathbf{x}^T(\sigma^T\mathbf{F}\sigma + \mathbf{D})\mathbf{x} \tag{3.9}$$

Proposition 3.4 *The unique optimal solution of problem (3.9) is*

$$\mathbf{x} = \frac{\lambda}{2(1-\lambda)}(\sigma^T\mathbf{F}\sigma + \mathbf{D})^{-1}(\mu - r_f\mathbf{I}_n),$$

and the maximum of (3.9) is

$$\lambda r_f + \frac{\lambda^2}{4(1-\lambda)}(\mu - r_f\mathbf{I}_n)^T(\sigma^T\mathbf{F}\sigma + \mathbf{D})^{-1}(\mu - r_f\mathbf{I}_n)$$

Proof Considering that matrix $(\sigma^T\mathbf{F}\sigma + \mathbf{D})$ and $(\sigma^T\mathbf{F}\sigma + \mathbf{D})^{-1}$ are positive definite and the objective function of problem (3.9) is strictly concave, there must exist unique optimal solution. The optimal solution and value can be obtained by Kohn-Tuck's one order condition. □

With the help of Proposition 3.2, 3.3 and 3.4, we can solve problem (2.8) via solving the following problem

$$\underset{(\mu,\sigma,\mathbf{D})\in\Pi}{Min}(\mu - r_f\mathbf{I}_n)^T(\sigma^T\mathbf{F}\sigma + \mathbf{D})^{-1}(\mu - r_f\mathbf{I}_n) \tag{3.10}$$

Proposition 3.5 *Problem (3.10) is equivalent to problem*:

$$\underset{\mu\in S_\mu}{Min}(\mu - r_f\mathbf{I}_n)^T(\sigma^{*T}\mathbf{F}\sigma^* + \bar{\mathbf{D}})^{-1}(\mu - r_f\mathbf{I}_n), \tag{3.11}$$

where $\bar{\mathbf{D}} = \mathbf{D}_0 + diag(\boldsymbol{\eta})$, $\boldsymbol{\sigma}_j^* = \boldsymbol{\sigma}_{0j} + \theta_{2j}^* \boldsymbol{\rho}_j$,

$$\theta_{2j}^* = \begin{cases} 0, & \boldsymbol{\rho}_j^T \mathbf{F} \boldsymbol{\sigma}_{0j} > 0; \\ 1, & \dfrac{\boldsymbol{\rho}_j^T \mathbf{F} \boldsymbol{\sigma}_{0j}}{\boldsymbol{\rho}_j^T \mathbf{F} \boldsymbol{\rho}_j^T} < -1; \\ -\dfrac{\boldsymbol{\rho}_j^T \mathbf{F} \boldsymbol{\sigma}_{0j}}{\boldsymbol{\rho}_j^T \mathbf{F} \boldsymbol{\rho}_j^T}, & \text{otherwise} \end{cases} \tag{3.12}$$

Proof For any $\mathbf{D} \in S_d, \mathbf{D} = \mathbf{D}_0 + diag(\boldsymbol{\theta}_3) diag(\boldsymbol{\eta})$, $\boldsymbol{\sigma} \in S_{\sigma,} \boldsymbol{\sigma}_j = \boldsymbol{\sigma}_{0j} + \boldsymbol{\theta}_{2j} \boldsymbol{\rho}_j$, $j = 1, 2, \ldots, n$, we first prove the inequality $\mathbf{y}^T(\boldsymbol{\sigma}^T \mathbf{F} \boldsymbol{\sigma} + \mathbf{D})\mathbf{y} \leq \mathbf{y}^T(\boldsymbol{\sigma}^{*T} \mathbf{F} \boldsymbol{\sigma}^* + \bar{\mathbf{D}})\mathbf{y}$ comes into existence for any $\mathbf{y} \in R^n$.

For any $\mathbf{y} \in R^n$, consider problem (3.13)

$$\underset{\boldsymbol{\theta}_2 \in (0,1)^n}{Max} \mathbf{y}^T(\boldsymbol{\sigma}^T \mathbf{F} \boldsymbol{\sigma})\mathbf{y}. \tag{3.13}$$

The Kuhn–Tucker conditions of problem (3.13) can be described as follows:

$$\begin{cases} 2y^T \boldsymbol{\rho}_j^T \mathbf{F} \boldsymbol{\sigma}_{0j} \mathbf{y} + 2\theta_{2j} y^T \boldsymbol{\rho}_j^T \mathbf{F} \boldsymbol{\rho}_j \mathbf{y} + u_j - v_j = 0 \\ u_j(1 - \theta_{2j}) = 0 \\ v_j \theta_{2j} = 0 \\ u_j > 0, \quad v_j > 0, \ j = 1, 2, \ldots, n \end{cases} \tag{3.14}$$

Formula (3.14) yields $\boldsymbol{\theta}_2^*$ in formula (3.12) when $\mathbf{y}^j = (0, \ldots, 1, \ldots, 0)^T$, $j = 1, 2, \ldots, n$.

Therefore, $\mathbf{y}^T(\boldsymbol{\sigma}^T \mathbf{F} \boldsymbol{\sigma})\mathbf{y} \leq \mathbf{y}^T(\boldsymbol{\sigma}^{*T} \mathbf{F} \boldsymbol{\sigma}^*)\mathbf{y}$ for any $\mathbf{y} \in R^n$ when $\boldsymbol{\theta}_2$ satisfies formula (3.12). In view of denotation of $\bar{\mathbf{D}}$, we can obtain inequality $\mathbf{y}^T(\boldsymbol{\sigma}^T \mathbf{F} \boldsymbol{\sigma} + \mathbf{D})\mathbf{y} \leq \mathbf{y}^T(\boldsymbol{\sigma}^{*T} \mathbf{F} \boldsymbol{\sigma}^* + \bar{\mathbf{D}})\mathbf{y}$ comes into existence for any $\mathbf{y} \in R^n$.

The eigenvalues of matrix $\boldsymbol{\sigma}^T \mathbf{F} \boldsymbol{\sigma} + \mathbf{D}$ and matrix $\boldsymbol{\sigma}^{*T} \mathbf{F} \boldsymbol{\sigma}^* + \bar{\mathbf{D}}$ are denoted λ_{1i}, $i = 1, 2, \ldots, n$ and λ_{2i}, $i = 1, 2, \ldots, n$ respectively. Considering that the two matrices above are positive definite, we can assume that $\lambda_{11} \geq \lambda_{12} \geq \cdots \geq \lambda_{1n} > 0$ and $\lambda_{21} \geq \lambda_{22} \geq \cdots \geq \lambda_{2n} > 0$ without loss of generality. Thus, we have $\lambda_{1i} \leq \lambda_{2i}$, $i = 1, 2, \ldots, n$. The values λ_{1i}^{-1}, $i = 1, 2, \ldots, n$ and λ_{2i}^{-1}, $i = 1, 2, \ldots, n$ are the eigenvalues of matrix $(\boldsymbol{\sigma}^T \mathbf{F} \boldsymbol{\sigma} + \mathbf{D})^{-1}$ and matrix $(\boldsymbol{\sigma}^{*T} \mathbf{F} \boldsymbol{\sigma}^* + \bar{\mathbf{D}})^{-1}$ and they satisfy $\lambda_{1i}^{-1} \geq \lambda_{2i}^{-1}$, $i = 1, 2, \ldots, n$. Therefore, $\mathbf{y}^T(\boldsymbol{\sigma}^T \mathbf{F} \boldsymbol{\sigma} + \mathbf{D})^{-1}\mathbf{y} \geq \mathbf{y}^T(\boldsymbol{\sigma}^{*T} \mathbf{F} \boldsymbol{\sigma}^* + \bar{\mathbf{D}})^{-1}\mathbf{y}$, for any $\mathbf{y} \in R^n$. This completes the proof. □

Proposition 3.6 *The optimization problem* (3.11) *has a unique optimal solution, denoted by* $\boldsymbol{\mu}^* = (\mu_1^*, \mu_2^*, \ldots, \mu_n^*)$.

Proof It is immediate since matrix $(\boldsymbol{\sigma}^{*T} \mathbf{F} \boldsymbol{\sigma}^* + \bar{\mathbf{D}})^{-1}$ is positive definite and the constraint set is nonempty compact. □

Remark 3.1 Considering that problem (3.11) is a quadratic programming problem, its optimal solution can be obtained by the simplex method solving linear programming problem.

According to Proposition 3.4, 3.5 and 3.6, the optimal solutions of problem (2.8) are

$$\mathbf{x} = \frac{\lambda}{2(1-\lambda)}(\boldsymbol{\sigma}^{*T}\mathbf{F}\boldsymbol{\sigma}^* + \bar{\mathbf{D}})^{-1}(\boldsymbol{\mu}^* - r_f\mathbf{I}_n). \tag{3.15}$$

The mean and variance of the returns are

$$M(\lambda) = r_f + \frac{\lambda}{2(\lambda-1)}(\boldsymbol{\mu}^* - r_f\mathbf{I}_n)^T(\boldsymbol{\sigma}^{*T}\mathbf{F}\boldsymbol{\sigma}^* + \bar{\mathbf{D}})^{-1}(\boldsymbol{\mu}^* - r_f\mathbf{I}_n), \tag{3.16}$$

$$V(\lambda) = \frac{\lambda^2}{4(1-\lambda)^2}(\boldsymbol{\mu}^* - r_f\mathbf{I}_n)^T(\boldsymbol{\sigma}^{*T}\mathbf{F}\boldsymbol{\sigma}^* + \bar{\mathbf{D}})^{-1}(\boldsymbol{\mu}^* - r_f\mathbf{I}_n). \tag{3.17}$$

Eliminating the parameter λ from (3.16) and (3.17), we can get the following explicit formula of the efficient frontier:

$$V = \frac{(M-r_f)^2}{(\boldsymbol{\mu}^* - r_f\mathbf{I}_n)^T(\boldsymbol{\sigma}^{*T}\mathbf{F}\boldsymbol{\sigma}^* + \bar{\mathbf{D}})^{-1}(\boldsymbol{\mu}^* - r_f\mathbf{I}_n)}. \tag{3.18}$$

In uncertainty set Π, the vector of mean returns, matrix of factor loadings and covariance matrix of residual returns are all vary in an interval. Formula (3.18) shows that in our robust model the efficient frontier does not depend on all covariance matrixes of residual returns in their interval except their upper bounds. However, the efficient frontier is related with the intervals of the vector of mean returns and matrix of factor loadings. These results may be interesting and important in reality.

4 Equilibrium price system

Along the lines of Konno and Shirakawa (1996) and Deng et al. (2005), in this section we discuss the existence and uniqueness and the explicit expression of an equilibrium price system $(p_1, p_2, \dots p_n)^T$ under which the market is cleared, i.e., the total demand matches the total supply of each asset.

Let us further assume that in the capital market. The total number of shares of asset j is y_j^0, $j = 1, 2, \dots, n+1$. There are k investors $i = 1, 2, \dots, k$ with the following characteristics: all investors have a planning horizon of one period and use the robust portfolio selection model; investors $i = 1, 2, \dots, k$ have the same estimated intervals for the mean returns vector, the factor loading matrix, the covariance matrix of the factor return vector, the residual covariance matrix and the same risk-less return; investor i invests an initial wealth W_i^0 in an initial portfolio $(y_{i1}^0, y_{i2}^0, \dots, y_{in+1}^0)$; investor i has a risk aversion factor $\lambda_i \in (0, 1]$.

Using the result in Sect. 3, each investor i will hold the asset portfolio

$$y_{ij}^* = \frac{W_i^0}{p_j}\frac{\lambda_i}{2(1-\lambda_i)}z_j, \quad j = 1, 2, \dots, n, \tag{4.1}$$

$$y_{in+1}^{*} = W_i^0 \left(1 - \frac{\lambda_i}{2(1-\lambda_i)} \sum_{j=1}^{n} z_j \right), \tag{4.2}$$

where z_j is the jth component of vector $(\boldsymbol{\sigma}^{*T}\mathbf{F}\boldsymbol{\sigma}^{*} + \bar{\mathbf{D}})^{-1}(\boldsymbol{\mu}^{*} - r_f \mathbf{I}_n)$. To clear the market, it must hold that the equilibrium prices $p_j,\ j = 1, 2, \ldots, n$, satisfy the linear system of equations

$$\sum_{i=1}^{k} \frac{\lambda_i}{2(1-\lambda_i)} \left(\sum_{l=1}^{n} p_l y_{il}^0 + y_{in+1}^0 \right) z_j = p_j y_j^0,\ \ j = 1, 2, \ldots, n. \tag{4.3}$$

Let

$$\alpha = \sum_{i=1}^{k} \sum_{j=1}^{n} \frac{\lambda_i}{2(1-\lambda_i)} \frac{y_{ij}^0}{y_j^0} z_j.$$

Proposition 4.1 *If $\alpha \neq 1$, then the system of Eq.* (4.3) *has an unique solution*

$$p_j = \frac{1}{1-\alpha} \frac{z_j}{y_j^0} \sum_{i=1}^{k} \frac{\lambda_i}{2(1-\lambda_i)} y_{in+1}^0,\ \ j = 1, 2, \ldots, n. \tag{4.4}$$

In particular, if we assume that $\sum_{i=1}^{k} \frac{\lambda_i}{2(1-\lambda_i)} y_{ij}^0 \geq 0$ for all $j = 1, 2, \ldots, n+1$ and $z_j \geq 0$ for all $j = 1, 2, \ldots, n$, then if and only if $\alpha < 1$ the Eq. (4.3) *has an unique nonnegative solution.*

5 Conclusions

It is common that the information about the distribution of an uncertainty variable is exactly unknown except that its mean and covariance are known varying over an interval in practice. The optimal solutions are often sensitive to perturbations in parameters of the optimization problem model constructed from the mean and covariance. To investigate the portfolio optimization problem, this paper proposes a robust approach maximizing worst-case utility when both the distributions underlying the uncertain vector of returns are exactly unknown and the estimates of the structure of returns are unreliable. We introduce concave convex utility function measuring the utility of investors under model uncertainty and uncertainty structure describing the moments of returns and all possible distributions and show that the robust portfolio optimization problem corresponding to the uncertainty structure can be reformulated as a parametric quadratic programming problem, enabling to obtain explicit formula solutions, an efficient frontier and equilibrium price system. This approach is an alternative of solving the optimal portfolios for an uncertainty-averse investor who takes a conservative viewpoint and identify asset mixes that have the best worst-case expected utility.

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